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M5-brane effective action as an on-shell action in supergravity

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Abstract: We show that the covariant effective action for M5-brane is a solution to the Hamilton-Jacobi (H-J) equations of 11-dimensional supergravity. The solution to the H-J equations reproduces the supergravity solution that represents the M2-M5 bound states.

Keywords: M-Theory, D-branes, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

D-branes and M-branes have been playing a crucial role in analyzing nonperturbative aspects of string theory (M-theory). D-branes can be regarded in string perturbation theory as a boundary state with the Dirichlet boundary condition imposed while they also emerge as classical solutions of supergravity.

In a series of publications [1, 2, 3], we showed that the D-brane effective action is a solution to the Hamilton-Jacobi (H-J) equations of type-IIA (IIB) supergravity and that the M2-brane and M5-brane effective actions are solutions to the H-J equations of 11-dimensional supergravity. We also showed that these solutions to the H-J equations reproduce the supergravity solutions which represent a stack of D-branes in a B_2 field, a stack of M2-branes and a stack of the M2-M5 bound states, respectively. This fact means that those effective actions of branes are on-shell actions around the corresponding brane solutions in supergravity. The near-horizon limit of these supergravity solutions are conjectured to be dual to various gauge theories [4, 5]. For instance, the near-horizon limit of the supergravity solution that represents D3-branes in a B_2 field is conjectured to be dual to noncommutative super Yang Mills in four dimensions [6, 7]. In gauge/gravity correspondence, the on-shell action around the background dual to a gauge theory is in general a generating functional of correlation functions in the gauge theory and the zero-mode part of the on-shell action should reproduce the holographic renormalization flow [8]. Hence,

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our findings should be useful for the study of the gauge/gravity (string) correspondence. (For other applications of our results, see the introduction in [2].)

In this paper, we revisit the M5-brane case. The near-horizon limit of the supergravity solution representing the M2-M5 bound states is conjectured to be dual to (a noncommutative version of) 6-dimensional $\mathcal{N} = (2,0)$ superconformal gauge field theory [4, 7]. There are two versions [11, 12] of the M5-brane effective action, which are equivalent in the sense that both give the same equations of motion for M5-brane [9]. One version of the action was suggested in [10] and explicitly constructed in [11]. In this version, in order to obtain a complete set of the equations of motion for M5-brane, one needs to add the self-duality condition to the equations obtained by varying the action. In fact, we showed in [2] that this action is a solution to the H-J equations up to the self-duality condition. This implies that this M5-brane effective action is not an on-shell in the ordinary sense. So, it seems unclear what role this effective action as a solutions to the H-J equations plays in the gauge/gravity correspondence. On the other hand, the other version of the action which was constructed in [12] and is called the 'covariant action' directly gives a complete set of the equations of motion for M5-brane. Although it contains an auxiliary scalar field that causes a problem in defining the partition function [10], it is well-defined at least at classical level. Thus, for the sake of the study of the gauge/gravity correspondence, it is worth investigating whether this covariant action satisfies the H-J equations of 11-dimensional supergravity and reproduces the supergravity solution representing the M2-M5 bound states so that the action is an on-shell action around the M2-M5 bound state solution in the ordinary sense. In this paper, we show that this is indeed the case.

The present paper is organized as follows. In section 2, we perform a reduction of 11-dimensional supergravity on S^4 and obtain a 7-dimensional gravity. In section 3, we develop the canonical formalism for the 7-dimensional gravity to derive the H-J equations. In section 4, we write down the covariant action for M5-brane explicitly. In section 5, we show that the covariant effective action is a solution to the H-J equations obtained in section 3. In section 6, we show that the solution to the H-J equations in the previous section reproduces the supergravity solution representing a stack of the M2-M5 bound states. In section 7, by using the relation of 11-dimensional supergravity with type-IIA supergravity, we obtain an effective action for NS 5-branes that is a solution to the H-J equations of type-IIA supergravity and reproduces the supergravity solution representing a stack of NS 5-branes, which is relevant for the duality between gravity and little string theory [15]. Section 8 is devoted to discussion. We give an argument which is expected to intuitively explain why the D-brane effective action satisfies the H-J equations of supergravity. In appendix, we present some equations which are useful for the calculations in sections 5 and 6.

2. Reduction of 11-dimensional supergravity on S^4

In this section, we perform a reduction of 11-dimensional supergravity on S^4 and obtain a 7-dimensional gravity. We will regard a radial-time-fixed surface in the 7-dimensional gravity as a worldvolume of M5-brane.

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The bosonic part of 11-dimensional supergravity is given by

$$I_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}X\sqrt{-G}\left(R_G - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int A_3\wedge F_4\wedge F_4\,, \tag{2.1}$$

where

$$F_4 = dA_3 \tag{2.2}$$

and

$$|F_4|^2 = \frac{1}{4!}F_{M_1M_2M_3M_4}F^{M_1M_2M_3M_4}\,. \tag{2.3}$$

We drop the fermionic degrees of freedom consistently. The equations of motion derived from (2.1) are

$$\begin{aligned} R^G_{MN} - \frac{1}{12}F_{ML_1L_2L_3}F_N{}^{L_1L_2L_3} + G_{MN}\left(-\frac{1}{2}R_G + \frac{1}{4}|F_4|^2\right) &= 0\,,\\ D_LF^{LM_1M_2M_3} - \frac{1}{2(4!)^2}\varepsilon^{M_1M_2M_3L_1\cdots L_8}F_{L_1L_2L_3L_4}F_{L_5L_6L_7L_8} &= 0\,, \end{aligned} \tag{2.4}$$

where D_M stands for the covariant derivative in eleven dimensions, while the Bianchi identity which follows from (2.2) is

$$dF_4 = 0\,. \tag{2.5}$$

We split the 11-dimensional coordinates X^M ($M = 0, 1, \ldots, 10$) into two parts as $X^M = (\xi^\alpha, \theta_i)$ ($\alpha = 0, \ldots, 6$, $i = 1, \ldots, 4$), where the ξ^α are 7-dimensional coordinates and the θ_i parametrize S^4. We make an ansatz for the fields as follows:

$$\begin{aligned} ds_{11} &= h_{\alpha\beta}(\xi)d\xi^\alpha d\xi^\beta + e^{\frac{\rho(\xi)}{2}}d\Omega_4\,, \qquad (2.6)\\ F_4 &= \frac{1}{4!}F_{\alpha_1\cdots\alpha_4}(\xi)d\xi^{\alpha_1}\wedge\cdots\wedge d\xi^{\alpha_4} + \frac{1}{4!\,7!}e^{\rho(\xi)}\varepsilon^{\alpha_1\cdots\alpha_7}\tilde{F}_{\alpha_1\cdots\alpha_7}(\xi)\varepsilon_{\theta_{i_1}\cdots\theta_{i_4}}d\theta_{i_1}\wedge\cdots\wedge d\theta_{i_4}\,. \end{aligned}$$

where $h_{\alpha\beta}$ is a 7-dimensional metric. Note that this ansatz preserves the 7-dimensional general covariance.

By substituting (2.6) into the equations of motion (2.4) and the Bianchi identity (2.5), we obtain the following equations in seven dimensions.

$$\begin{aligned} R_{\alpha\beta} - \nabla_\alpha\nabla_\beta\rho - \frac{1}{4}\partial_\alpha\rho\partial_\beta\rho - \frac{1}{12}F_{\alpha\gamma_1\gamma_2\gamma_3}F_\beta{}^{\gamma_1\gamma_2\gamma_3} - &\\ -\frac{1}{2}h_{\alpha\beta}\left(R + e^{-\frac{\rho}{2}}R^{(S^4)} - 2\nabla_\gamma\nabla^\gamma\rho - \frac{5}{4}\partial_\gamma\rho\partial^\gamma\rho - \frac{1}{2}|F_4|^2 + \frac{1}{2}|\tilde{F}_7|^2\right) &= 0\,,\\ R + \frac{1}{2}e^{-\frac{1}{2}\rho}R^{(S^4)} - \frac{3}{2}\nabla_\gamma\nabla^\gamma\rho - \frac{3}{4}\partial_\gamma\rho\partial^\gamma\rho - \frac{1}{2}|F_4|^2 - \frac{1}{2}|\tilde{F}_7|^2 &= 0\,,\\ \nabla_\gamma(e^\rho F^{\gamma\alpha_1\alpha_2\alpha_3}) + \frac{1}{4!}e^\rho F_{\gamma_1\gamma_2\gamma_3\gamma_4}\tilde{F}^{\alpha_1\alpha_2\alpha_3\gamma_1\gamma_2\gamma_3\gamma_4} &= 0\,,\\ \varepsilon^{\alpha\beta\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5}\partial_{\gamma_1}F_{\gamma_2\gamma_3\gamma_4\gamma_5} &= 0\,,\\ \nabla_\gamma(e^\rho\tilde{F}^{\gamma\alpha_1\cdots\alpha_6}) &= 0\,, \end{aligned} \tag{2.7}$$

where ∇_α stands for the covariant derivative in seven dimensions, and $R^{(S^4)} = 12$. By using a relation in seven dimensions,

$$\tilde{F}_{\alpha\gamma_1\cdots\gamma_6}\tilde{F}_\beta{}^{\gamma_1\cdots\gamma_6} = \frac{1}{7}h_{\alpha\beta}\tilde{F}_{\gamma_1\cdots\gamma_7}\tilde{F}^{\gamma_1\cdots\gamma_7}\,,$$

we can check that these equations are derived from the 7-dimensional gravity given by

$$I_7 = \int d^7\xi\sqrt{-h}\,e^\rho\left(R_h + e^{-\frac{\rho}{2}}R^{(S^4)} + \frac{3}{4}\partial_\alpha\rho\,\partial^\alpha\rho - \frac{1}{2}|F_4|^2 - \frac{1}{2}|\tilde{F}_7|^2\right), \tag{2.8}$$

where

$$\begin{aligned} F_4 &= dA_3\,,\\ \tilde{F}_7 &= dA_6 - \frac{1}{2}A_3\wedge F_4\,. \end{aligned} \tag{2.9}$$

This reduction is a consistent truncation in the sense that every solution of I_7 can be lifted to a solution of 11-dimensional supergravity.

3. Canonical formalism and the H-J equations

In this section, we develop the canonical formalism for I_7 obtained in the previous section and derive the H-J equations. First, we rename the 7-dimensional coordinates:

$$\xi^\mu = x^\mu \quad (\mu = 0,\cdots,5)\,, \qquad \xi^6 = r\,.$$

Adopting r as time, we make the ADM decomposition for the 7-dimensional metric.

$$\begin{aligned} ds_7^2 &= h_{\alpha\beta}\,d\xi^\alpha d\xi^\beta\\ &= (n^2 + g^{\mu\nu}n_\mu n_\nu)\,dr^2 + 2n_\mu\,dr\,dx^\mu + g_{\mu\nu}\,dx^\mu dx^\nu\,, \end{aligned} \tag{3.1}$$

where n and n_μ are the lapse function and the shift function, respectively. Henceforce μ, ν run from 0 to 5.

In what follows, we consider a boundary surface specified by $r =$ const. and impose the Dirichlet condition for the fields on the boundary. Here we need to add the Gibbons-Hawking term [16] to the actions, which is defined on the boundary and ensures that the Dirichlet condition can be imposed consistently. Then, the 7-dimensional action I_7 with the Gibbons-Hawking term on the boundary can be expressed in the canonical form as follows:

$$\begin{aligned} I_7 = \int dr d^6x\sqrt{-g}\Big(&\pi^{\mu\nu}\partial_r g_{\mu\nu} + \pi_\rho\partial_r\rho + \pi^{\mu_1\mu_2\mu_3}\partial_r A_{\mu_1\mu_2\mu3} + \pi^{\mu_1\cdots\mu_6}\partial_r A_{\mu_1\cdots\mu_6} -\\ &- nH - n_\mu H^\mu - A_{r\mu\nu}G_2^{\mu\nu} - A_{r\mu_1\cdots\mu_5}G_5^{\mu_1\cdots\mu_5}\Big) \end{aligned} \tag{3.2}$$

with

$$H = e^{-\rho}\bigg(- (\pi^{\mu\nu})^2 + \frac{1}{9}(\pi^\mu{}_\mu)^2 - \frac{5}{9}\pi_\rho^2 + \frac{4}{9}\pi^\mu{}_\mu\pi_\rho -$$

$$
\begin{aligned}
&\quad - 3(\pi^{\mu_1\mu_2\mu_3} + 10\pi^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}A_{\nu_1\nu_2\nu_3})^2 - \frac{6!}{2}(\pi^{\mu_1\cdots\mu_6})^2\Big) - \mathcal{L}\,,\\
H^\mu &= -2\nabla^g_\nu \pi^{\mu\nu} + \partial^\mu\rho\pi_\rho + F^\mu{}_{\nu_1\nu_2\nu_3}\pi^{\nu_1\nu_2\nu_3} + \left(F^\mu{}_{\nu_1\cdots\nu_6} - \frac{15}{2}A^\mu{}_{\nu_1\nu_2}F_{\nu_3\cdots\nu_6}\right)\pi^{\nu_1\cdots\nu_6}\,,\\
G_2^{\mu\nu} &= -3\nabla^g_\lambda \pi^{\lambda\mu\nu}\,,\\
G_5^{\mu_1\cdots\mu_5} &= -6\nabla^g_\rho \pi^{\rho\mu_1\cdots\mu_5}\,, \qquad (3.3)
\end{aligned}
$$

where

$$
\mathcal{L} = e^\rho\left(R_g - 2\nabla^g_\mu\nabla^\mu_g\rho - \frac{5}{4}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}|F_4|^2 - \frac{1}{2}|\tilde{F}_7|^2\right) + e^{\frac{\rho}{2}}R^{(S^4)}\,, \qquad (3.4)
$$

$\pi^{\mu\nu}$ is the canonical momentum conjugate to $g_{\mu\nu}$, and so on. The relations between the canonical momenta and the r derivatives of the fields are given by

$$
\begin{aligned}
\pi_{\mu\nu} &= e^\rho\left(-K_{\mu\nu} + g_{\mu\nu}K + \frac{1}{n}g_{\mu\nu}(\partial_r\rho - n^\lambda\partial_\lambda\rho)\right),\\
\pi_\rho &= e^\rho\left(2K + \frac{3}{2}\frac{1}{n}(\partial_r\rho - n^\mu\partial_\mu\rho)\right),\\
\pi_{\mu_1\mu_2\mu_3} &= e^\rho\bigg(- \frac{1}{6}\frac{1}{n}(F_{r\mu_1\mu_2\mu_3} - n^\nu F_{\nu\mu_1\mu_2\mu_3}) +\\
&\qquad + \frac{1}{72}\frac{1}{n}(\tilde{F}_{r\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} - n^\nu\tilde{F}_{\nu\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3})A^{\nu_1\nu_2\nu_3}\bigg),\\
\pi_{\mu_1\cdots\mu_6} &= -\frac{1}{6!}e^\rho\frac{1}{n}(\tilde{F}_{r\mu_1\cdots\mu_6} - n^\nu\tilde{F}_{\nu\mu_1\cdots\mu_6})\,, \qquad (3.5)
\end{aligned}
$$

where $K_{\mu\nu}$ is the extrinsic curvature given by

$$
K_{\mu\nu} = \frac{1}{2n}(\partial_r g_{\mu\nu} - \nabla^g_\mu n_\nu - \nabla^g_\nu n_\mu)\,, \qquad K = g^{\mu\nu}K_{\mu\nu}\,. \qquad (3.6)
$$

n, n_μ and $A_{r\mu\nu}$ and $A_{r\mu_1\cdots\mu_5}$ behave like Lagrange multipliers and give the constraints:

$$
H = 0\,, \qquad H^\mu = 0\,, \qquad G_2^{\mu\nu} = 0 \qquad \text{and} \qquad G_5^{\mu_1\cdots\mu_5} = 0\,. \qquad (3.7)
$$

The first one and the second one are called the hamiltonian constraint and the momentum constraint respectively, while the third one and the last one are called the Gauss law constraints. Note that the hamiltonian density, $\mathcal{H} = nH + n_\mu H^\mu + A_{r\mu\nu}G_2^{\mu\nu} + A_{r\mu_1\cdots\mu_5}G_5^{\mu_1\cdots\mu_5}$, vanishes on shell due to these constraints.

As usual, the H-J equation is obtained by performing the following replacements in the hamiltonian.

$$
\begin{aligned}
\pi^{\mu\nu}(x) &= \frac{1}{\sqrt{-g(x)}}\frac{\delta S}{\delta g_{\mu\nu}(x)}\,, \qquad \pi_\rho(x) = \frac{1}{\sqrt{-g(x)}}\frac{\delta S}{\delta\rho(x)}\,,\\
\pi^{\mu_1\mu_2\mu_3}(x) &= \frac{1}{\sqrt{-g(x)}}\frac{\delta S}{\delta A_{\mu_1\mu_2\mu_3}(x)}\,, \qquad \pi^{\mu_1\cdots\mu_6}(x) = \frac{1}{\sqrt{-g(x)}}\frac{\delta S}{\delta A_{\mu_1\cdots\mu_6}(x)}\,, \qquad (3.8)
\end{aligned}
$$

where S is an on-shell action, and $g_{\mu\nu}(x)$, $\rho(x)$, $A_{\mu_1\mu_2\mu_3}(x)$ and $A_{\mu_1\cdots\mu_6}(x)$ represent the values of the fields on the boundary $r =$ const.. The fact that the hamiltonian vanishes on shell simplifies the ordinary H-J equation:

$$\frac{\partial S}{\partial r}+\int d^6x\mathcal{H}=0\ \rightarrow\ \frac{\partial S}{\partial r}=0\,. \tag{3.9}$$

This implies that S does not depend on the boundary 'time' r explicitly but depend only on the boundary values of the fields. In addition to the ordinary H-J equation (3.9), there are a set of equations for S which is obtained by applying the replacements (3.8) to the constraints (3.7). These equations should also be called the H-J equations. For instance, the H-J equation coming from $H^\mu=0$ takes the form

$$\begin{aligned}-2\nabla^g_\nu\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)+\partial^\mu\rho\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\rho}+F^\mu_{\ \nu_1\nu_2\nu_3}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_{\nu_1\nu_2\nu_3}}+&\\+\left(F^\mu_{\ \nu_1\cdots\nu_6}-\frac{15}{2}A^\mu_{\ \nu_1\nu_2}F_{\nu_3\cdots\nu_6}\right)\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_{\nu_1\cdots\nu_6}}&=0\,.\end{aligned} \tag{3.10}$$

The H-J equations coming from $H^\mu=0$, $G_2^{\mu\nu}=0$ and $G_5^{\mu_1\cdots\mu_5}=0$ gives a condition that S must be invariant under the diffeomorphism in six dimensions and the U(1) gauge transformations

$$\begin{aligned}A_3&\rightarrow A_3+d\Sigma_2\,,\\A_6&\rightarrow A_6+d\Sigma_5+\frac{1}{2}\Sigma_2\wedge F_4\,.\end{aligned} \tag{3.11}$$

(See appendix C in ref. [1]). The H-J equation coming from $H=0$ is a nontrivial equation that can determine the form of S.

4. Covariant M5-brane action

Before solving the H-J equations obtained in the previous section, we write down the covariant effective action for M5-brane, which was constructed in [12].[1] It takes the form

$$S_{M5}=-\int d^6\sigma\left(\sqrt{-\det(\mathcal{G}_{\mu\nu}+\tilde{\mathcal{H}}_{\mu\nu})}+\frac{1}{4}\sqrt{-\mathcal{G}}\check{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}_{\mu\nu}\right)+\int d^6\sigma\left(\mathcal{A}_6+\frac{1}{2}\mathcal{A}_3\wedge F_3\right), \tag{4.1}$$

where the σ^μ ($\mu=0,\ldots,5$) parametrize the worldvolume of the M5-brane and $\mathcal{G}_{\mu\nu}$, $\mathcal{A}_3$ and $\mathcal{A}_6$ are the induced fields on the worldvolume of the corresponding fields in 11-dimensional supergravity. For instance, $\mathcal{G}_{\mu\nu}$ is given by

$$\mathcal{G}_{\mu\nu}(\sigma)=\frac{\partial Y^M(\sigma)}{\partial\sigma^\mu}\frac{\partial Y^N(\sigma)}{\partial\sigma^\nu}G_{MN}(Y(\sigma))\,, \tag{4.2}$$

where the $Y^M(\sigma)$ ($M=0,\ldots,10$) are embedding functions of the worldvolume in eleven dimensions. F_3 is the gauge field strength on the worldvolume, which is a third-rank antisymmetric tensor. There also exists an auxiliary scalar field on the worldvolume, which is denoted by a. It is convenient to introduce a time-like unit vector field v_μ:

$$v_\mu=\frac{c_\mu}{\sqrt{-c_\nu c^\nu}}\,,\qquad c_\mu=\frac{\partial a}{\partial\sigma^\mu}\,,\qquad v_\mu v^\mu=-1\,. \tag{4.3}$$

[1] For a canonical formulation and a gauge fixing for the covariant M5-brane action, see ref. [17].

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Then $\check{\mathcal{H}}_{\mu\nu}$ and $\tilde{\mathcal{H}}_{\mu\nu}$ are defined in terms of $\mathcal{G}_{\mu\nu}$, $\mathcal{A}_3$, F_3 and v_μ as follows:

$$\begin{aligned}
\mathcal{H}_{\mu\nu\lambda} &= \mathcal{A}_{\mu\nu\lambda} + F_{\mu\nu\lambda}\,, \\
\mathcal{H}^{*\mu\nu\lambda} &= \frac{1}{6}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}\mathcal{H}_{\rho\sigma\tau}\,, \\
\check{\mathcal{H}}_{\mu\nu} &= \mathcal{H}_{\mu\nu\lambda}v^\lambda\,, \\
\tilde{\mathcal{H}}_{\mu\nu} &= \mathcal{H}^*_{\mu\nu\lambda}v^\lambda\,.
\end{aligned} \tag{4.4}$$

Note that the effective action (4.1) is the lowest order in derivative expansion so that the effective action is valid only when the fields are almost independent of σ.

5. Covariant M5-brane action as a solution to the H-J equations

In solving the H-J equations of the 7-dimensional gravity, we drop the dependence of the fields on the 6-dimensional coordinates x^μ. Correspondingly, any solution of supergravity obtained from such a solution to the H-J equations will depend only on the radial time r. In other words, we reduce the problem to a one-dimensional one. Let S_0 be a solution to the H-J equations under this simplification. It follows from (3.3), (3.4) and (3.8) that the H-J equations coming from the hamiltonian constraint $H=0$ is simplified as

$$\begin{aligned}
&\left(\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta g_{\mu\nu}}\right)^2 - \frac{1}{9}\left(g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta g_{\mu\nu}}\right)^2 + \frac{5}{9}\left(\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta \rho}\right)^2 - \\
&-\frac{4}{9}g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta g_{\mu\nu}}\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta\rho} + 3\left(\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta A_{\mu\nu\lambda}} + 10A_{\rho_1\rho_2\rho_3}\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta A_{\mu\nu\lambda\rho_1\rho_2\rho_3}}\right)^2 + \\
&+\frac{6!}{2}\left(\frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta A_{\mu_1\cdots\mu_6}}\right)^2 + e^{\frac{3}{2}\rho}R^{(S^4)} = 0\,.
\end{aligned} \tag{5.1}$$

Let us consider the following form:

$$S_0 = S_c + S_{BI} + S_{WZ}\,, \tag{5.2}$$

with

$$\begin{aligned}
S_c &= \alpha\int d^6x\sqrt{-g}e^{\frac{3}{4}\rho}\,, \\
S_{BI} &= \beta\int d^6x\left(\sqrt{-\det(g_{\mu\nu}+\tilde{H}_{\mu\nu})} + \frac{1}{4}\sqrt{-g}\check{H}^{\mu\nu}\tilde{H}_{\mu\nu}\right), \\
S_{WZ} &= \gamma\int d^6x\left(A_6 + \frac{1}{2}A_3\wedge F_3\right),
\end{aligned} \tag{5.3}$$

where $\check{H}_{\mu\nu}$ and $\tilde{H}_{\mu\nu}$ are defined in terms of $g_{\mu\nu}$, A_3, F_3 and v_μ in the same way as (4.4),

$$\begin{aligned}
H_{\mu\nu\lambda} &= A_{\mu\nu\lambda} + F_{\mu\nu\lambda}\,, \\
H^{*\mu\nu\lambda} &= \frac{1}{6}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}H_{\rho\sigma\tau}\,, \\
\check{H}_{\mu\nu} &= H_{\mu\nu\lambda}v^\lambda\,, \\
\tilde{H}_{\mu\nu} &= H^*_{\mu\nu\lambda}v^\lambda\,,
\end{aligned} \tag{5.4}$$

and v_μ is defined in terms of c_μ in the same way as (4.3). All of the fields in (5.3) are independent of x^μ so that the integral over the 6-dimensional space-time is factored out. In what follows, we show that S_0 (5.2) is a solution to the simplified H-J equations of the 7-dimensional gravity with c_μ and F_3 being arbitrary constants if

$$\alpha^2 = \frac{16}{3} R^{(S^4)} = 64 \qquad \text{and} \qquad \beta = -\gamma\,. \tag{5.5}$$

S_0 trivially satisfies the simplified H-J equations of the 7-dimensional gravity except (5.1), while S_0 satisfies (5.1) quite nontrivially, as we will see below.

If (5.2) is substituted into (5.1), the lefthand side of (5.1) is decomposed into four parts:

$$\text{l.h.s. of (5.1)} = (1) + (2) + (3) + (4) \tag{5.6}$$

with

$$\begin{aligned}
(1) &= \left(\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta g_{\mu\nu}}\right)^2 - \frac{1}{9}\left(g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta g_{\mu\nu}}\right)^2 + \frac{5}{9}\left(\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta \rho}\right)^2 - \\
&\quad -\frac{4}{9} g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta g_{\mu\nu}}\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta \rho} + e^{\frac{3}{2}\rho} R^{(S^4)}\,, \\
(2) &= 2 g_{\mu\lambda} g_{\nu\rho}\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta g_{\mu\nu}}\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\lambda\rho}} - \frac{2}{9} g_{\mu\nu} g_{\lambda\rho}\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta g_{\mu\nu}}\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\lambda\rho}} - \\
&\quad -\frac{4}{9} g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta \rho}\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\mu\nu}}\,, \\
(3) &= \left(\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\mu\nu}}\right)^2 - \frac{1}{9}\left(g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\mu\nu}}\right)^2 + \\
&\quad +3\left(\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta A_{\mu\nu\lambda}} + \frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda}} + 10 A_{\rho\sigma\tau}\frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda\rho\sigma\tau}}\right)^2, \\
(4) &= \frac{6!}{2}\left(\frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda\rho\sigma\tau}}\right)^2.
\end{aligned} \tag{5.7}$$

By using the relations

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta g_{\mu\nu}} &= \frac{1}{2}\alpha e^{\frac{3}{4}\rho} g^{\mu\nu}\,, \\
\frac{1}{\sqrt{-g}}\frac{\delta S_c}{\delta \rho} &= \frac{3}{4}\alpha e^{\frac{3}{4}\rho}\,,
\end{aligned} \tag{5.8}$$

we can easily calculate (1), (2). The results are

$$\begin{aligned}
(1) &= -\frac{3}{16}\alpha^2 e^{\frac{3}{2}\rho} + e^{\frac{3}{2}\rho} R^{(S^4)}\,, \\
(2) &= \left(1 - \frac{2}{3} - \frac{1}{3}\right) g_{\mu\nu}\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\mu\nu}} = 0\,.
\end{aligned} \tag{5.9}$$

If the first condition in (5.5) holds, (1) also vanishes. By using

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda\rho\sigma\tau}} = \frac{\gamma}{6!}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}\,, \tag{5.10}$$

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we can also calculate (4) easily:

$$(4) = -\frac{\gamma^2}{2}\,. \tag{5.11}$$

The calculation of (3) is a nontrivial task. In appendix, we present some equations which are useful for the calculation of (3). By using those equations and the second condition in (5.5), we obtain

$$(3) = \frac{\gamma^2}{2}\,. \tag{5.12}$$

Thus (3) cancels (4). Therefore, (5.1) is actually satisfied with $F_{\mu\nu\lambda}$ and c_μ being arbitrary constants.

In the remaining of this section, we show that our solution (5.2) can be interpreted as the covariant M5 action (4.1). As is clear from (4.2), if one sets in (4.1)

$$Y^\mu(\sigma) = \sigma^\mu\,, \qquad \text{other } Y^M = 0\,, \qquad \sigma^\mu = x^\mu\,, \tag{5.13}$$

the induced fields in (4.1) reduce to the fields in the 7-dimensional gravity:

$$\mathcal{G}_{\mu\nu}(\sigma) = g_{\mu\nu}(x)\,, \qquad \mathcal{A}_{\mu\nu\lambda}(\sigma) = A_{\mu\nu\lambda}(x)\,, \qquad \mathcal{A}_{\mu_1\cdots\mu_6}(\sigma) = A_{\mu_1\cdots\mu_6}(x)\,. \tag{5.14}$$

Hence, if F_3 and c_μ in (5.3) are also identified with those in (4.1) and the σ-dependence of the fields in (4.1) are completely neglected,

$$S_{BI} + S_{WZ} = -\beta\, S_{M5}\,. \tag{5.15}$$

In other words, $S_{BI} + S_{WZ}$ is the effective action for M5-brane whose worldvolume is not curved in the transverse directions. However, as in ref. [3], we can show by taking into account the freedom of general coordinate transformation that the effective action for M5-brane whose worldvolume is curved in the r-direction is also a solution to the H-J equations (up to a term analogous to S_c in (5.2)). Whether the effective action for M5-brane with a general configuration is a solution to the H-J equations of 11-dimensional supergravity is an open problem. Finally, we make a comment on S_c. We found in [1, 2] that the D-brane effective action with a term analogous to S_c is a solution to the H-J equations of supergravity. The term analogous to S_c has a dependence on the dilaton field ϕ like $e^{-2\phi}$ while the D-brane effective action has a dependence like $e^{-\phi}$ if the R-R fields are rescaled appropriately. Therefore, the term analogous to S_c should correspond to a contribution from the sphere amplitude in string theory. (Of course, the D-brane effective action is a contribution from the disk amplitude.) Thus, S_c should correspond to a contribution from M-theory counterpart of the sphere amplitude in string theory.

6. Supergravity solution of the M2-M5 bound state

The supergravity solution that represents a stack of N M2-M5 bound states is given in [18], and it is also a solution of I_7 which takes the following form:

$$\begin{aligned} ds_7^2 &= f^{-\frac{2}{3}} k^{\frac{1}{3}} \eta_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\mu}} + f^{\frac{1}{3}} k^{-\frac{2}{3}} \delta_{ab} dx^a dx^b + f^{\frac{1}{3}} k^{\frac{1}{3}} dr^2\,, \\ e^{\frac{\varrho}{2}} &= r^2 f^{\frac{1}{3}} k^{\frac{1}{3}}\,, \qquad A_{012} = \sin\theta f^{-1}\,, \qquad A_{345} = \tan\theta k^{-1}\,, \\ \tilde{F}_{012345r} &= 3\cos\theta \tilde{Q} r^{-4} f^{-1} k^{-1}\,, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned}
\hat{\mu},\, \hat{\nu} &= 0,1,2\,, \qquad a,\, b = 3,4,5\,, \\
f &= 1+\frac{\tilde{Q}}{r^3}\,, \qquad \tilde{Q} = \frac{\pi N}{\cos\theta}\,, \qquad k = \sin^2\theta + \cos^2\theta f\,,
\end{aligned} \tag{6.2}$$

θ is a parameter of the solution. Note that this solution preserves sixteen supercharges and that when $\theta = 0$ the solution reduces to the ordinary solution representing a stack of N M5-branes.

Now that we obtained a solution (5.2) to the H-J equations in the previous section, we have a set of first-order differential equations, which can be regarded as an integral of the equations of motion:

$$\begin{aligned}
\pi^{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta g_{\mu\nu}}\,, \qquad \pi_\rho = \frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta\rho}\,, \\
\pi^{\mu\nu\lambda} &= \frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta A_{\mu\nu\lambda}}\,, \qquad \pi^{\mu\nu\lambda\rho\sigma\tau} = \frac{1}{\sqrt{-g}}\frac{\delta S_0}{\delta A_{\mu\nu\lambda\rho\sigma\tau}}\,,
\end{aligned} \tag{6.3}$$

where (3.5) is substituted into the lefthand sides while (5.2) into the righthand sides.

We evaluate the both sides of (6.3) by substituting the above solution (6.1). The results for the lefthand sides are

$$\begin{aligned}
\pi_{\mu\nu} &= \eta_{\mu\nu}\left(4r^3 k^{\frac{5}{6}} f^{-\frac{1}{6}} + \frac{1}{2} r^4 k^{\frac{5}{6}} f^{-\frac{7}{6}} \partial_r f\right), \\
\pi_{ab} &= \delta_{ab}\left(4r^3 f^{\frac{5}{6}} k^{-\frac{1}{6}} + \frac{1}{2} r^4 f^{\frac{5}{6}} k^{-\frac{7}{6}} \partial_r k\right), \\
\pi_\rho &= 6r^3 k^{\frac{1}{2}} f^{\frac{1}{2}}\,, \\
\pi_{012} &= -\frac{1}{4}\sin\theta\tilde{Q} k^{\frac{1}{2}} f^{-\frac{3}{2}}\,, \\
\pi_{345} &= -\frac{1}{4}\sin\theta\cos\theta\tilde{Q} k^{-\frac{3}{2}} f^{\frac{1}{2}}\,, \\
\pi_{012345} &= -\frac{3}{6!}\cos\theta\tilde{Q} f^{-\frac{1}{2}} k^{-\frac{1}{2}}\,,
\end{aligned} \tag{6.4}$$

where the other $\pi_{\mu\nu\lambda}$ and $\pi_{\mu\nu\lambda\rho\sigma\tau}$ vanish. We can also evaluate the righthand sides by using (5.8), (5.10) and the equations in appendix. The righthand sides coincide with the lefthand sides (6.4) if we set in S_0

$$\begin{aligned}
F_{\mu\nu\lambda} &= 0\,, \\
\alpha &= 8\,, \\
\beta &= -\gamma = -3\tilde{Q}\cos\theta\,.
\end{aligned} \tag{6.5}$$

The second and third equations are consistent with (5.5). Thus the supergravity solution (6.1) satisfies the first-order differential equations (6.3) given by S_0 (5.2). Hence, S_0 reproduces the supergravity solution (6.1). That is, S_0 is an on-shell action around the supergravity solution (6.1). Note that we need not put any restriction on c_μ in order for (6.1) to satisfy (6.3).

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7. NS 5-brane in type-IIA supergravity

In this section, using the relation between 11-d supergravity and type-IIA supergravity, we obtain a solution to the H-J equation of type-IIA supergravity that reproduces the supergravity solution representing a stack of NS 5-branes. So, this solution should correspond to a type-IIA NS 5-brane effective action (plus a S_c-like term). First, we consider a reduction of 11-d supergravity on $S^3 \times S^1$, which is different from the one done in section 2:

$$\begin{aligned} ds_{11}^2 &= G_{MN} dX^M dX^N \\ &= h_{\alpha\beta}(\xi) d\xi^\alpha d\xi^\beta + e^{\frac{1}{2}\rho_2(\xi)} d\Omega_3 + e^{\frac{1}{2}\rho_1(\xi)} (dX^{10})^2 \,, \end{aligned} \tag{7.1}$$

$$\begin{aligned} F_4 &= \frac{1}{4!} F_{\alpha_1 \cdots \alpha_4} d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_4} + \\ &+ \frac{1}{3!\,7!} e^{\frac{1}{4}\rho_1(\xi) + \frac{3}{4}\rho_2(\xi)} \varepsilon_{\alpha_1 \cdots \alpha_7} \tilde{F}^{\alpha_1 \cdots \alpha_7} \varepsilon_{\theta_{i_1}\theta_{i_2}\theta_{i_3}} d\theta_{i_1} \wedge d\theta_{i_2} \wedge d\theta_{i_3} \wedge dX^{10} \,, \end{aligned} \tag{7.2}$$

where α, β run from 0 to 6, and X^{10} parametrizes S^1. Then, we obtain as a consistent truncation a seven-dimensional gravity

$$I_7' = \int d^7\xi \sqrt{-h}\, e^{\frac{1}{4}\rho_1 + \frac{3}{4}\rho_2} \left(R_h + e^{-\frac{\rho_2}{2}} R^{(S^3)} + \frac{3}{8} \partial_\alpha \rho_1 \, \partial^\alpha \rho_2 + \frac{3}{8} \partial_\alpha \rho_2 \, \partial^\alpha \rho_2 - \frac{1}{2} |F_4|^2 - \frac{1}{2} |\tilde{F}_7|^2 \right) , \tag{7.3}$$

where $F_4 = dA_3$, $\tilde{F}_7 = dA_6 - \frac{1}{2} A_3 \wedge F_4$ and $R^{(S^3)} = 6$. Let us consider the form

$$S_0' = S_c' + S_{BI} + S_{WZ} \,, \tag{7.4}$$

with

$$S_c' = \tilde{\alpha} \int d^6x \sqrt{-g}\, e^{\frac{1}{4}\rho_1 + \frac{1}{2}\rho_2} \,, \tag{7.5}$$

and S_{BI} and S_{WZ} the same in (5.3). We can verify that (7.4) satisfies the H-J equation of I_7' under the same simplification as section 5 if

$$\tilde{\alpha}^2 = 6 R^{(S^3)} = 36 \,, \qquad \text{and} \qquad \beta = -\gamma \,. \tag{7.6}$$

Next, following the relation between 11-d supergravity and type-IIA supergravity, we define the fields in type-IIA supergravity in terms of those in 11-d supergravity as follows.

$$\begin{aligned} h_{\alpha\beta} &= e^{-\frac{2}{3}\phi} k_{\alpha\beta} \,, \\ \rho_1 &= \frac{8}{3}\phi \,, \qquad \rho_2 = \rho - \frac{4}{3}\phi \,, \\ A_3 &= -C_3 \,, \qquad A_6 = -B_6 \,. \end{aligned} \tag{7.7}$$

I_7' is rewritten in terms of these new fields as

$$\begin{aligned} I_7' = \int d^7\xi \sqrt{-k} \Big[& e^{-2\phi + \frac{3}{4}\rho} \left(R_k + e^{-\frac{1}{2}\rho} R^{(S^3)} + 4 \partial_\alpha \phi \partial^\alpha \phi + \frac{3}{8} \partial_\alpha \rho \partial^\alpha \rho - 3 \partial_\alpha \phi \partial^\alpha \rho \right) - \\ & - \frac{1}{2} e^{\frac{3}{4}\rho} |F_4^{(IIA)}|^2 - \frac{1}{2} e^{2\phi + \frac{3}{4}\rho} |\tilde{H}_7|^2 \Big] \,, \end{aligned} \tag{7.8}$$

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where $F_4^{(IIA)} = dC_3$ and $\tilde{H}_7 = dB_6 + \frac{1}{2}C_3 \wedge F_4$. This action is actually given by a consistent truncation of type-IIA supergravity in which the ansatz for the fields is

$$\begin{aligned}
ds_{10}^2 &= k_{\alpha\beta}(\xi)d\xi^\alpha d\xi^\beta + e^{\frac{1}{2}\rho(\xi)}d\Omega_3 \, , \\
\phi &= \phi(\xi) \, , \\
H_3 &= -\frac{1}{3!\,7!}e^{2\phi+\frac{3}{4}\rho}\varepsilon_{\alpha_1\cdots\alpha_7}\tilde{H}^{\alpha_1\cdots\alpha_7}(\xi)\varepsilon_{\theta_{i_1}\theta_{i_2}\theta_{i_3}}d\theta_{i_1}\wedge d\theta_{i_2}\wedge d\theta_{i_3} \, , \\
F_4^{(IIA)} &= \frac{1}{4!}F^{(IIA)}_{\alpha_1\cdots\alpha_4}(\xi)d\xi^{\alpha_1}\wedge\cdots\wedge d\xi^{\alpha_4} \, ,
\end{aligned} \tag{7.9}$$

where H_3 and $F_4^{(IIA)}$ are the NS-NS anti-symmetric field strength and the R-R field strength, respectively. Thus, by rewriting (7.4) in terms of the fields in (7.8), we obtain a solution to the H-J equations of (7.8):

$$S_0^{(NS5)} = S_c^{(NS5)} + S_{BI}^{(NS5)} + S_{WZ}^{(NS5)} \tag{7.10}$$

with

$$\begin{aligned}
S_c^{(NS5)} &= \tilde{\alpha}\int d^6x\sqrt{-g}e^{-2\phi+\frac{1}{2}\rho} \, , \\
S_{BI}^{(NS5)} &= \beta\int d^6x e^{-2\phi}\left(\sqrt{-\det(g_{\mu\nu}+e^\phi\tilde{H}^{(NS5)}_{\mu\nu})} + \frac{1}{4}\sqrt{-g}\check{H}^{(NS5)\mu\nu}\tilde{H}^{(NS5)}_{\mu\nu}\right) , \\
S_{WZ}^{(NS5)} &= \beta\int\left(B_6 + \frac{1}{2}C_3\wedge F_3\right) ,
\end{aligned} \tag{7.11}$$

where $g_{\mu\nu}$ is the (μ,ν) component of $k_{\alpha\beta}$, and $\check{H}^{(NS5)}_{\mu\nu}$ and $\tilde{H}^{(NS5)}_{\mu\nu}$ are defined in terms of $H^{(NS5)}_{\mu\nu\lambda} = -C_{\mu\nu\lambda} + F_{\mu\nu\lambda}$ and c_μ in the same way as (5.4). This solution to the H-J equations obviously reproduces the supergravity solution of type-IIA NS 5-brane and should correspond to the NS 5-brane effective action (plus $S_c^{(NS5)}$ term). In fact, up to $S_c^{(NS5)}$, the solution (7.10) coincides with β times the effective action for type-IIA NS 5-brane which is proposed in [14] if in the effective action the R-R 1-form and its partner 1-form on the worldvolume are put to zero.

8. Discussion

In this paper, we showed that the covariant effective action for M5-brane is an on-shell action around the solution in 11-dimensional supergravity that represents a stack of the M2-M5 bound states. Applying our result to the gauge/gravity correspondence is a future work.

In refs. [1, 2], we showed that the same thing holds for the D-brane case. That is, the D-brane effective action (the Born-Infeld action plus the Wess-Zumino action) is an on-shell action around the solution in type-IIA(IIB) supergravity representing a stack of D-branes. The following argument is expected to intuitively explain why the D-brane effective action satisfies the H-J equations of supergravity and is an on-shell action of supergravity. Suppose that there exists a string field theory for type-IIA or IIB superstring. There are two limits that can be taken for the string field theory. One is the low energy limit (the $\alpha' \to 0$ limit), and the other is the classical limit, $g_s \to 0$, where g_s is the string coupling. The string field theory would reduce to type-IIA or IIB supergravity in the $\alpha' \to 0$ limit while in $g_s \to 0$ limit it would reproduce the results of the lowest order in the string perturbation theory. As the string field theory is a quantum theory of gravity, wave functions or transition amplitudes in the string field theory should satisfy the equations analogous to the Wheeler-DeWitt (WDW) equations, which we call the WDW-like equations, where we regard the radial direction as time. In $\alpha' \to 0$, these equations reduce to the WDW equations of the corresponding supergravity. Moreover, in the classical limit ($g_s \to 0$ limit), the WDW equations reduce to the H-J equations in the corresponding supergravity, which we are concerned with. Let us reverse the order of these two limits. That is, when the $g_s \to 0$ limit is first taken in the string field theory, one obtains from the WDW-like equations the equations of string theory which are analogous to the H-J equations and contain all α' corrections. We call these equations the H-J-like equations. Note that it is difficult to write down these the H-J-like equations because the conformal or the light-cone gauges do not seem to fit deriving the equations. Next, these H-J-like equations reduce in the $\alpha' \to 0$ limit to the H-J equations in supergravity.

Let us consider a transition amplitude between the vacuum and a state that represents a stack of N D-branes in the string field theory. The amplitude must satisfy the WDW-like equations and reduces in $g_s \to 0$ limit to the transition amplitude between the vacuum and the D-brane boundary state, which must satisfy the H-J-like equations. Moreover, under a condition that the fields on the worldvolumes of the D-branes vary slowly, the amplitude is represented by the D-brane effective action (the Born-Infeld action plus the Wess-Zumino terms). That is, the D-brane effective action satisfies the H-J-like equations under the above condition. A nontrivial thing is that our results show that the D-brane effective action satisfies the H-J equations itself. Probably supersymmetry make this nontrivial thing possible.

Finally, let us check whether it is consistent that the solution to the H-J equations of supergravity actually represents the effective action for a stack of N D-branes. As in the M5-brane case (see (6.5)), the Born-Infeld action plus the Wess-Zumino terms in the solution is proportional to N when the solution is matched to the supergravity solution representing a stack of N D-branes. This is consistent with the fact that the tension of the stack of D-branes is proportional to N. The reason why the solution to the H-J equations does not possess the non-abelian property is as follows. Neglecting the dependence on the worldvolume coordinates in solving the H-J equations implies imposing $D_\mu F_{\nu\lambda} = 0$ due to the gauge invariance, where $F_{\mu\nu}$ is the gauge field strength on the worldvolume, so that from the Bianchi identity $[F_{\mu\nu}, F_{\lambda\rho}] = 0$. Therefore, we cannot see the non-abelian part of the gauge field strength.

In the near future, we hope to refine this argument and completely understand the reason why the D-brane effective action is a solution to the H-J equations of supergravity.

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A. Useful equations

In this appendix, we present some equations which are useful for the calculations of (3) in section 5 and of the canonical momenta in section 6. By using a relation $\det \tilde{H}_{\mu\nu} = 0$, which follows from $\tilde{H}_{\mu\nu} v^\nu = 0$, one obtains

$$\det(g_{\mu\nu} + \tilde{H}_{\mu\nu}) = g\left(1 - \frac{1}{2}\,\mathrm{tr}\,\tilde{H}^2 + \frac{1}{8}(\mathrm{tr}\,\tilde{H}^2)^2 - \frac{1}{4}\,\mathrm{tr}\,\tilde{H}^4\right) \equiv gE\,. \tag{A.1}$$

It is convenient to introduce an antisymmetric tensor $D_{\mu\nu}$:

$$D_{\mu\nu} \equiv -\sqrt{\frac{1}{E}}\left((1 - \frac{1}{2}\,\mathrm{tr}\,\tilde{H}^2)\tilde{H}_{\mu\nu} + \tilde{H}_{\mu\lambda}\tilde{H}^{\lambda\rho}\tilde{H}_{\rho\nu}\right). \tag{A.2}$$

In terms of E and $D_{\mu\nu}$, the derivatives of S_{BI} and S_{WZ} are expressed shortly:

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta g_{\mu\nu}} &= \beta\Big[-\frac{1}{4}v^\mu v^\nu \tilde{H}^{\lambda\sigma}\check{H}_{\lambda\sigma} - \frac{1}{8}\tilde{H}_{\lambda\sigma}(H^{\lambda\sigma\mu}v^\nu + H^{\lambda\sigma\nu}v^\mu) + \frac{1}{2}\sqrt{E}g^{\mu\nu} + \\
&\qquad + \frac{1}{2}D^\mu_{\ \lambda}\tilde{H}^{\lambda\nu} + \frac{1}{4}(g^{\mu\nu} + v^\mu v^\nu)D^{\lambda\sigma}\tilde{H}_{\lambda\sigma}\Big]\,,\\
\frac{1}{\sqrt{-g}}\frac{\delta S_{BI}}{\delta A_{\mu\nu\lambda}} &= \beta\left[\frac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}(2D_{\rho\sigma} - \check{H}_{\rho\sigma})v_\tau + \frac{1}{12}(\tilde{H}^{\mu\nu}v^\lambda + \tilde{H}^{\nu\lambda}v^\mu + \tilde{H}^{\lambda\mu}v^\nu)\right],\\
\frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda}} &= \frac{\gamma}{72}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}F_{\rho\sigma\tau}\,.
\end{aligned} \tag{A.3}$$

(5.10) and the last equation in (A.3) gives

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda}} + 10A_{\rho\sigma\tau}\frac{1}{\sqrt{-g}}\frac{\delta S_{WZ}}{\delta A_{\mu\nu\lambda\rho\sigma\tau}} = \frac{\gamma}{72}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}H_{\rho\sigma\tau}\,, \tag{A.4}$$

where the lefthand side is a combination appearing in (3).

In what follows, we evaluate the quantities needed in calculating the righthand side of (6.3). First, by substituting (6.1), we obtain

$$\begin{aligned}
\tilde{H}_{01} &= -\tan\theta k^{\frac{1}{6}}f^{-\frac{5}{6}}v_2\,, & \tilde{H}_{12} &= \tan\theta k^{\frac{1}{6}}f^{-\frac{5}{6}}v_0\,, & \tilde{H}_{20} &= -\tan\theta k^{\frac{1}{6}}f^{-\frac{5}{6}}v_1\,,\\
\tilde{H}_{34} &= -\sin\theta k^{-\frac{5}{6}}f^{\frac{1}{6}}v_5\,, & \tilde{H}_{45} &= -\sin\theta k^{-\frac{5}{6}}f^{\frac{1}{6}}v_3\,, & \tilde{H}_{53} &= -\sin\theta k^{-\frac{5}{6}}f^{\frac{1}{6}}v_4\,,\\
\check{H}_{01} &= \sin\theta k^{-\frac{1}{3}}f^{-\frac{1}{3}}v_2\,, & \check{H}_{12} &= -\sin\theta k^{-\frac{1}{3}}f^{-\frac{1}{3}}v_0\,, & \check{H}_{20} &= \sin\theta k^{-\frac{1}{3}}f^{-\frac{1}{3}}v_1\,,\\
\check{H}_{34} &= \tan\theta k^{-\frac{1}{3}}f^{-\frac{1}{3}}v_5\,, & \check{H}_{45} &= \tan\theta k^{-\frac{1}{3}}f^{-\frac{1}{3}}v_3\,, & \check{H}_{53} &= \tan\theta k^{-\frac{1}{3}}f^{-\frac{1}{3}}v_4\,.
\end{aligned} \tag{A.5}$$

Next, note that there is a relation between v_μ, which follows from $v_\mu v^\mu = -1$,

$$k^{-\frac{1}{3}}f^{\frac{2}{3}}t + k^{\frac{2}{3}}f^{-\frac{1}{3}}u = -1\,, \tag{A.6}$$

where

$$t = -v_0^2 + v_1^2 + v_2^2\,, \qquad u = v_3^2 + v_4^2 + v_5^2\,. \tag{A.7}$$

By using this relation and (A.5), E in (A.1) is calculated as

$$E = \frac{kf^{-1}}{\cos^2\theta}(1 + k^{-\frac{1}{3}} f^{-\frac{1}{3}} \sin^2\theta u)^2 \,. \tag{A.8}$$

From this, we obtain

$$D_{\mu\nu} = \check{H}_{\mu\nu} \tag{A.9}$$

when (6.1) is substituted.

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